Weak limits and their calculation in analog signal theory

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Abstract

Purpose – This paper aims to improve the mathematical justification of certain analog signal theory concepts and offer a rigorous framework for it.

Design/methodology/approach – The framework relies on functional analysis, namely theory of distributions and the concept of weak limit. Its notation is adjusted to resemble the notation usually used in engineering signal theory. It can be used to prove in a rigorous manner already established results in signal theory, but also to establish new ones.

Findings – Examples have shown the lack of rigour caused by using ordinary calculus in proving fundamental signal theoretic results. On that basis, concepts of limit, Fourier transform and derivative are revisited in the spirit of functional analysis. A new useful formula for weak limit computation is proved.

Originality/value – Functional analysis is efficiently used in signal theory in a manner approachable by engineers. An original and efficient formula for weak limit computation is presented and proved.

Keywords Analog signal theory, Functional analysis, Schwartz distribution, Weak limit, Fourier transforms

Paper type Research paper

1. Introduction

Signal theory is a branch of applied mathematics that uses various tools from advanced calculus to explore properties of various physical quantities described in abstract form that vary in time and/or space (signals). Another aim of signal theory is deriving relationships between responses of various abstract models of real physical processes (systems) to applied stimulus. If we assume that all quantities are known in any instance of time and/or space, then we talk about analog signal theory.

To achieve proposed goals, analog signal theory sometimes must use very advanced tools from functional analysis (especially from theory of distributions) and Fourier analysis. Unfortunately, although signal theory is developed mainly for applications in engineering, these advanced tools are often far beyond mathematical knowledge of a typical engineer. Consequently, in many prevalent courses of analog signal theory, such tools are presented in quite oversimplified form, which formally makes them look like well-known tools from ordinary calculus. Using such approach, the theory itself may be presented in a way that may be understandable even to non-experts in mathematics (Papoulis, 1962, 1977). In addition, such approach

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often produces final results which are valid in practice, even if intermediate steps which are performed in deriving these results are not rigorous (the validity of such steps is often quite suspicious). The main shortcoming of such approach is the fact that we have no clue under which circumstances derived results are valid, and what happens if they are not valid. Maybe the worst consequence of non-rigorous approach is the possibility that someone (a student or a potential researcher) might derive completely incorrect conclusions if any of non-rigorous steps is performed without very special precautions (which are not given in the simplified theory). Of course, the same results may be derived in a rigorous manner using very advanced mathematics (Vladimirov, 1979; Reed and Simon, 1980; Rudin, 1973), but we already said that the rigorous derivation is incomprehensible for most users of signal theory.

In this paper, we will discuss the concept of limit, which is often misinterpreted in analog signal theory. Namely, the classic concept of limit known from ordinary calculus is often inadequate for usage in analog signal theory and, as we shall see, sometimes may produce incorrect results. Another concept of limit, so-called weak limit (or, even more precise, weak-star limit), which is more adequate for usage in analog signal theory, is known in advanced mathematics for years. For example, this concept may be found (in various forms) in publications by Hadamard, Fréchet, Banach, Riesz and Sobol'ev. Today, it is one of the fundamental concepts in functional analysis. Unfortunately, weak limit is usually defined using very abstract concepts, which look completely strange to an average engineer (Reed and Simon, 1980; Rudin, 1973). In this paper, we will show how it is possible to introduce weak limits and work with them using the terminology and notation that is not far from the terminology and notation known from ordinary calculus, and which is usually used in analog signal theory. Of course, such approach is significantly better than ignoring the concept of weak limit completely (which is usual approach in teaching analog signal theory).

Unfortunately, no definition of weak limit known from functional analysis gives any useful method for calculating weak limits. That is why the weak limit of a weakly convergent process in functional analysis is usually anticipated. Afterwards, various advanced tools are used to prove that the anticipated result is correct. Such approach is completely inadequate for usage in signal theory. So, the main contribution of this paper is a formula that allows to determine the weak limit in a lot of situations which arise very often in the signal theory. The proof of the formula is also given in the paper. The formula itself is quite useful and it is understandable enough to be included in more advanced courses dedicated to analog signal theory. The usefulness of the given formula is also demonstrated in the paper.

After this introduction, Section 2 of the paper deals with the concept of weak limit, introducing it and comparing it with the ordinary limit. Section 3 introduces the new method for weak limit calculation, appropriate for engineering purposes. Section 4 uses the weak limit concept for adding rigor in familiar concepts like derivatives and Fourier transform. After the conclusions, an Appendix is given with proofs of theorems stated in the paper.

2. The concept of the weak limit

A careful reader who reads any book about analog signal theory may notice that the classic concept of limit known from calculus cannot be applied to some formulae that are used in the signal theory. Probably the most obvious example occurs in the theory of Fourier transform. There are many formulae related to Fourier transform (Papoulis, 1962, 1977) that cannot be explained using the classic concept of limit. For example, many books give the following formula:

$$\mathcal{F}\{\operatorname{sgn} t\} = -i\frac{2}{\omega} \tag{1}$$

Deriving this formula indirectly is quite easy (Papoulis, 1977). However, if we try to derive this formula directly from the usual definition of Fourier transform, we will find a lot of troubles:

$$\begin{aligned}\mathcal{F}\{\operatorname{sgn} t\} &= \int_{-\infty}^{\infty} \operatorname{sgn} t e^{-i\omega t} dt = -\int_{-\infty}^{0} e^{-i\omega t} dt + \int_{0}^{\infty} e^{-i\omega t} dt = -\int_{0}^{\infty} e^{i\omega t} dt + \int_{0}^{\infty} e^{-i\omega t} dt \\ &= -\int_{0}^{\infty} (e^{i\omega t} - e^{-i\omega t}) dt = -2i\int_{0}^{\infty} \sin \omega t \, dt = 2i\frac{\cos \omega t}{\omega}\Big|_{t=0}^{t=\infty} = 2i\left(\lim_{t\to\infty}\frac{\cos \omega t}{\omega} - \frac{1}{\omega}\right)\end{aligned} \tag{2}$$

This result reduces to equation (1) only if we can somehow justify the following limit:

$$\lim_{t\to\infty}\frac{\cos \omega t}{\omega} = 0 \tag{3}$$

This is, of course, impossible using classic definition of limit known from the calculus (note that equation (3) should be valid in some sense even for $\omega = 0$).

The second example that suggests that equation (3) might be correct in some generalized sense comes from studying behavior of real systems in the time domain. Assume that we applied signal $f(t) = \sin \omega t$ as an input of a real linear time-invariant system. As we know from the signal theory, the response $g(t)$ will be again a sine wave (probably shifted) with amplitude $A(\omega)$, where $A(\omega)$ is the magnitude of a frequency transfer function of the system. Of course, $A(\omega)$ depends of ω. However, almost all real systems attenuate strongly very high frequencies, so $A(\omega) \to 0$ as $\omega \to \infty$. So, the response $g(t)$ tends to zero as $\omega \to \infty$. This means that real-world systems sense a rapid sine wave as a zero input. As this is true for all real-world linear time-invariant systems, we can conclude that the following relation must be valid in some sense, although it is completely meaningless using classic definition of limit:

$$\lim_{\omega\to\infty} \sin \omega t = 0 \tag{4}$$

Obviously, equation (4) is closely related with equation (3).

Someone might object that the assumption $f(t) = \sin \omega t$ is quite unrealistic, because $f(t)$ is an eternal wave. Nevertheless, it can be shown that the same conclusion can be derived for causal sine wave, i.e. for $f(t) = \sin \omega t \, u(t)$, where $u(t)$ is Heaviside step function. For example, if we apply $f(t)$ as an input to first order system (say, RC circuit where $\tau = RC$), we can derive the following response:

$$g(t) = \int_{0}^{t} \sin \omega\xi e^{-\tau(t-\xi)} d\xi = \frac{1}{\omega^2 + \tau^2}[\omega e^{-\tau t} + \tau \sin \omega t - \omega \cos \omega t]u(t) \tag{5}$$

Obviously, this response also tends to zero as ω tends to infinity.

Another closely related example is given by Feynman *et al.* (1964). In order to calculate the total radiation field produced by a sheet of oscillating sources in a point at a large, finite distance from the sheet, integral in question was reduced to:

$$\int_{r=z}^{r=\infty} e^{-i\omega r/c} \mathrm{d}r = -\frac{c}{i\omega}\left[e^{-i\infty} - e^{-(i\omega/c)z}\right] \tag{6}$$

(notation quoted without any editing). Feynman *et al.* (1964) writes:

> Now $e^{-i\infty}$ is a mysterious quantity. Its real part, for example, is cos $(-\infty)$, which, mathematically speaking, is completely indefinite (although we would expect it to be somewhere – or everywhere (?) – between $+1$ and -1!). But in a physical situation, it can mean something quite reasonable, and usually can just be taken to be zero.

Again, physical reality indicates that mathematical tools being used are not exactly what is needed.

These examples show that the concept of limit should be generalized for proper usage in the signal theory. Moreover, the following example shows that the concept of limit for purposes of signal theory must be completely changed, not only generalized. Namely, in previous examples, the classical limit does not exist at all. In the following example, the classical (pointwise) limit of applied input sequence exists, but real-world linear systems have completely different sensation about what this limit should be. Assume that we applied the following sequence of signals as stimulus of some real-world linear time-invariant system:

$$f_n(t) = 2n^{\alpha} t e^{-n^2t^2} u(t) \tag{7}$$

Here, α is a positive real parameter. This sequence converges to zero when n tends to infinity for all fixed values of t, so in classical (pointwise) sense, we obviously have:

$$\lim_{n\to\infty} f_n(t) = 0 \tag{8}$$

Now, we will show that the most of real-world linear time-invariant systems do not share the same opinion. Studying the general case might be quite difficult. So, let us assume for the beginning that a sequence of stimuli $f_n(t)$ is applied to, say, pure integrator. Then, we will get the following sequence of responses:

$$g_n(t) = \int_0^t f_n(\xi)\mathrm{d}\xi = n^{\alpha-2}(1 - e^{-n^2t^2})u(t) \tag{9}$$

The behavior of this sequence depends strongly of the value of α. For $\alpha < 2$, $g_n(t)$ really tends to zero as $n \to \infty$, but for $\alpha \geq 2$, we have completely different behavior. For $\alpha > 2$, $g_n(t)$ diverges, and for $\alpha = 2$, $g_n(t)$ tends to the step function $u(t)$. We can derive the similar conclusion for many other linear time-invariant systems. For example, if we replace the pure integrator with the first-order system (let us take $\tau = 1$ due to simplicity), we will get:

$$g_n(t)=\int_0^t f_n(\xi)e^{\xi-t}\mathrm{d}\xi=n^{\alpha-2}\left[e^{-t}-e^{-n^2t^2}+\frac{\sqrt{\pi}}{2n}e^{(1/4n^2)-t}\left(\mathrm{erf}\frac{2n^2t-1}{2n}-\mathrm{erf}\frac{1}{2n}\right)\right]u(t) \tag{10}$$

Again, we see that $g_n(t)$ converges to zero for $\alpha < 2$, diverges for $\alpha > 2$, and converges to $e^{-t}u(t)$ for $\alpha = 2$. If we look carefully, we will see that $u(t)$ is the impulse response of the pure integrator, and that $e^{-t}u(t)$ is the impulse response of the first-order system. Obviously, this cannot be just a coincidence: $f_n(t)$ really in some sense, which we want to explain, converges to impulse (but only for $\alpha = 2$).

Now, we will formalize the concept of so-called weak limit. In reality, any conclusion about real behavior of a physical process may be derived only using measurements. However, any measurement is, in fact, a response of some system – measurement device, which, at least theoretically, should be linear and time-invariant. So, it seems natural to define that the family of stimuli $f_\lambda(t)$ tends to $f(t)$ when $\lambda \to \lambda_0$ if and only if for each linear time-invariant system $\mathrm{L}[\cdot]$ the sequence of responses $g_\lambda(t) = \mathrm{L}[f_\lambda(t)]$ converges to response $g(t) = \mathrm{L}[f(t)]$ when $\lambda \to \lambda_0$. This is exactly what is called weak-limit: $f_\lambda(t) \to f(t)$ weakly when $\lambda \to \lambda_0$ if and only if $\mathrm{L}[f_\lambda(t)] \to \mathrm{L}[f(t)]$ for each linear time-invariant operator L taken from some class of such operators (Vladimirov, 1979; Reed and Simon, 1980; Rudin, 1973; Schwartz, 1965; Antosik *et al.*, 1973). If we express operator L using the convolution integral, we can give the alternative definition: $f_\lambda(t) \to f(t)$ weakly when $\lambda \to \lambda_0$ if and only if for each function $\phi(t)$ taken from some class of functions the following relation is satisfied:

$$\lim_{\lambda\to\lambda_0}\int_{-\infty}^{\infty} f_\lambda(t)\phi(t)\mathrm{d}t = \int_{-\infty}^{\infty} f(t)\phi(t)\mathrm{d}t \tag{11}$$

The most natural question is which class of functions $\phi(t)$ we should consider. Functional analysis defines various types of weak limits, depending just on the class of functions $\phi(t)$ for which equation (11) must hold. Let us consider which class is the most appropriate class for the applications in the signal theory. If we assume that the measurement process is causal, and if we pick the result of measurement in a finite time, it is easy to conclude that $\phi(t)$ is always time-limited function (i.e. function with compact support). In the theory of generalized functions (Vladimirov, 1979; Schwartz, 1965; Antosik *et al.*, 1973), it is required that equation (11) must be satisfied for each time-limited and infinitely smooth function $\phi(t)$ (such functions are called test functions, and its space is called $\mathcal{D}(\mathbb{R})$). Restriction to the smooth functions $\phi(t)$ makes some generalizations much easier. Note that this restriction is not too restrictive even for practical considerations, because in reality parasitic effects prevent instantaneous changes in physical quantities during the measurement process, so $\phi(t)$ is always smooth in reality. Therefore, we will accept the requirement that equation (11) must hold for each test function $\phi(t)$.

In fact, it is easy to see that the weak limit cannot be determined completely uniquely. Namely, if $f(t)$ is a weak limit of $f_\lambda(t)$ when $\lambda \to \lambda_0$ and if $\sigma(t)$ is any function which is equal to zero everywhere except on a zero measure set (so-called null function), then $f(t) + \sigma(t)$ is also a weak limit of $f_\lambda(t)$ when $\lambda \to \lambda_0$. Also, if $f^*(t)$ and $f^{**}(t)$ are two different weak limits of $f_\lambda(t)$ when $\lambda \to \lambda_0$, then $f^*(t) - f^{**}(t)$ is necessarily a null function (Reed and Simon, 1980; Rudin, 1973; Schwartz, 1965; Natanson, 1957).

Therefore, the weak limit can be determined uniquely only up to a null function, i.e. almost everywhere. To make the definition of the weak limit less indeterminate, we will accept the convention that the "correct" weak limit is one that is continuous at as many points as possible. Under such convention, the weak limit is determined uniquely at each point of continuity. At the points of discontinuity, the weak limit may be defined arbitrarily, for example, at the point $t = 0$ when $f(t) = u(t)$.

Using given definition of weak limit and applying Riemann-Lebesgue lemma known from Fourier analysis (Vladimirov, 1979; Reed and Simon, 1980; Natanson, 1957), it may be shown quite easily that:

$$\mathrm{w.}\lim_{\omega\to\infty} \sin \omega t = 0 \tag{12}$$

This is exactly what we concluded using naive reasoning. Here, the notation w.lim is used to make distinction between ordinary and weak limit. This distinction is necessary, because we saw that ordinary and weak limits do not always coincide. Unfortunately, notation w.lim is rarely used in engineering literature, although nearly all limits in analog signal theory are in fact weak limits. This may be misleading, especially because some formulae that are true for classical limits are not true for weak limits. For example, the following well-known formula is valid for classical limits (assuming that both sides of this formula exist):

$$\lim_{\lambda\to\lambda_0} f_\lambda(t) g_\lambda(t) = \lim_{\lambda\to\lambda_0} f_\lambda(t) \cdot \lim_{\lambda\to\lambda_0} g_\lambda(t) \tag{13}$$

However, this formula often fails with weak limits. Here is a simple counterexample:

$$\mathrm{w.}\lim_{\omega\to\infty} (\sin \omega t \cdot \sin \omega t) = \mathrm{w.}\lim_{\omega\to\infty} \frac{1 - \cos 2\omega t}{2} = \frac{1}{2} - \frac{1}{2}\mathrm{w.}\lim_{\omega\to\infty} \cos 2\omega t = \frac{1}{2} \tag{14}$$

$$\mathrm{w.}\lim_{\omega\to\infty} \sin \omega t \cdot \mathrm{w.}\lim_{\omega\to\infty} \sin \omega t = 0 \cdot 0 = 0$$

More specifically, as the weak limit is a product of the linear functional analysis, we can expect that it may behave pathologically in combination with non-linear operations (like multiplication).

The following theorem gives a simple sufficient condition for equality of pointwise and weak limit.

Theorem 1. Let $f_\lambda(t)$ be a family of locally integrable real functions (i.e. real functions such that their integral is finite on each finite interval). If there exists a locally integrable function $f(t)$ such that $|f_\lambda(t)| \leq f(t)$ for all values of λ from some neighborhood of $\lambda = \lambda_0$ (finite or infinite, where neighborhood of an infinite value $+\infty$ or $-\infty$ is any set that contains interval $(c, +\infty)$ or $(-\infty, c)$ where c is some constant) and if $f_\lambda(t)$ converges pointwise everywhere except maybe on a zero measure set, then:

$$\mathrm{w.}\lim_{\lambda\to\lambda_0} f_\lambda(t) = \lim_{\lambda\to\lambda_0} f_\lambda(t) \tag{15}$$

This theorem is a direct consequence of Lebesgue convergence theorem known from real analysis. Its proof may be found in Natanson (1957) and Royden (1988), where it is given in slightly different but equivalent form.

Note that the conditions of Theorem 1 are not satisfied for the sequence $f_n(t)$ given by equation (7), and we saw that for this sequence pointwise and weak limit does not coincide. Let us apply equation (11) to find what is the weak limit of $f_n(t)$ for $\alpha = 2$:

$$\begin{aligned}\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(t)\phi(t)\mathrm{d}t &= \lim_{n\to\infty} 2n^2\int_0^{\infty} te^{-n^2t^2}\phi(t)\mathrm{d}t\\ &= \lim_{n\to\infty} 2n^2\int_0^{1/\sqrt{n}} te^{-n^2t^2}\phi(t)\mathrm{d}t + \lim_{n\to\infty} 2n^2\int_{1/\sqrt{n}}^{\infty} te^{-n^2t^2}\phi(t)\mathrm{d}t\end{aligned} \tag{16}$$

We can make the following estimation for the second integral:

$$\left|2n^2\int_{1/\sqrt{n}}^{\infty} te^{-n^2t^2}\phi(t)\mathrm{d}t\right| \le 2n^2\int_{1/\sqrt{n}}^{\infty} te^{-n^2t^2}|\phi(t)|\mathrm{d}t \le 2n^2\mathrm{M}\int_{1/\sqrt{n}}^{\infty} te^{-n^2t^2}\mathrm{d}t = \mathrm{M}e^{-n} \tag{17}$$

Here, $\mathrm{M} = \max|\phi(t)|$. From this estimation, we conclude that the second integral tends to zero as $n \to \infty$. The first integral can be transformed using the Mean Value Theorem, so we have (here, $0 < \vartheta < 1$):

$$\begin{aligned}\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(t)\phi(t)\mathrm{d}t &= \lim_{n\to\infty} 2n^2\phi\left(\frac{\vartheta}{\sqrt{n}}\right)\int_0^{1/\sqrt{n}} te^{-n^2t^2}\mathrm{d}t\\ &= \lim_{n\to\infty}\phi\left(\frac{\vartheta}{\sqrt{n}}\right)(1-e^{-n}) = \phi(0)\end{aligned} \tag{18}$$

Therefore, if the weak limit of sequence $f_n(t)$ is equal to some function $f(t)$, then $f(t)$ must satisfy:

$$\int_{-\infty}^{\infty} f(t)\phi(t)\mathrm{d}t = \phi(0) \tag{19}$$

Strictly speaking, it is not hard to prove that such function $f(t)$ does not exist (Vladimirov, 1979; Reed and Simon, 1980; Rudin, 1973). However, we can recognize that equation (19) is just the basic property of the well-known generalized impulse function $\delta(t)$, expressed using the notation which is usually seen in engineering books (Papoulis, 1962, 1977). Therefore, if we allow that the weak limit may be a generalized function, we can say that:

$$\mathrm{w.}\lim_{n\to\infty} f_n(t) = \delta(t) \tag{20}$$

This proves what we already concluded intuitively before. If we want to be mathematically rigorous, then equation (11) should be rewritten in a form of generalized inner product (Vladimirov, 1979), i.e. as:

$$\lim_{\lambda\to\lambda_0}\langle f_\lambda(t), \phi(t)\rangle = \langle f(t), \phi(t)\rangle \tag{21}$$

Namely, the inner product is well defined even for generalized functions, which is not always true for the integral. In this paper, we will use the convention that generalized functions are just Schwartz distributions (although there are also other kinds of generalized functions). We will recall that the Schwartz distribution $f(t)$ is a purely symbolic object (not a function of a real argument t, regardless of the notation) which acts as a functional (operator) that assigns to each test function $\phi(t)$ a numeric value which is by definition taken as a value of the inner product $\langle f(t), \phi(t)\rangle$ (Vladimirov, 1979; Reed and Simon, 1980; Rudin, 1973; Schwartz, 1965). For example, the generalized impulse function $\delta(t)$ is in fact the operator that assigns to each test function $\phi(t)$ the value $\phi(0)$, which is by definition the value of the inner product $\langle\delta(t), \phi(t)\rangle$. If we want to be as rigorous as possible, it is additionally requested that $f(t)$ as an operator must be continuous in respect to appropriately constructed topology (Vladimirov, 1979; Reed and Simon, 1980; Rudin, 1973), but this is not so important for this paper. Note that each classical locally integrable function $f(t)$ may be also interpreted in distributional sense as a functional that assigns to each test function $\phi(t)$ the value:

$$\langle f(t), \phi(t)\rangle = \int_{-\infty}^{\infty} f(t)\phi(t)\mathrm{d}t \tag{22}$$

If $f(t)$ as an operator may be extended to space of functions $\phi(t)$ such that they together with all their derivatives decay more rapidly than any polynomial when t tends to infinity although their support is not necessarily compact, then we say that $f(t)$ is a tempered distribution. Sufficient condition for ordinary locally integrable function $f(t)$ to be tempered is that there exists a polynomial $P(t)$ such that $f(t)/P(t)$ is absolutely integrable on $\mathbb{R}$.

It is important to notice that it is not possible to deduce what is the weak limit of a sequence of functions just by simple superficial observation. For example, look at the following picture, which shows the sequence $f_n(t)$ given by equation (7) for three different values of α (Figure 1).

As we already deduced, sequence $f_n(t)$ converges to zero (both ordinarily and weakly) for $\alpha < 2$, converges (weakly) to $\delta(t)$ for $\alpha = 2$, and weakly diverges for $\alpha > 2$. Therefore, from the system aspect of view, the sequence in the left picture is a zero sequence, the sequence in the picture in the middle is an impulse sequence, and the sequence in the right picture is a divergent sequence. Obviously, this cannot be deduced just by observation of these pictures.

3. The proposed method for calculation of weak limits

The weak limit is a fairly well-known concept in advanced mathematics (Vladimirov, 1979; Reed and Simon, 1980; Rudin, 1973; Schwartz, 1965; Antosik, 1973). Unfortunately, this important concept may be very difficult to calculate by usual means. From the reasoning given in the previous section, we can easily conclude that finding weak limit

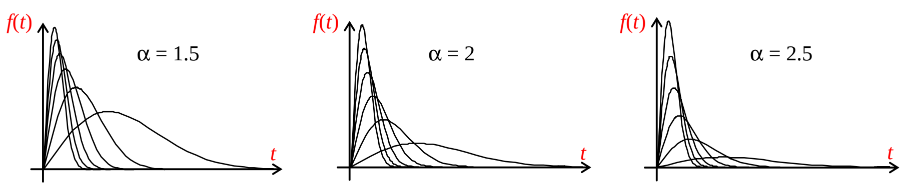

Figure 1.
Sequence (7) for different values of the parameter α

may be very complicated task. For example, we saw that finding weak limit of sequence $f_n(t)$ given by equation (7) required some tricky manipulations. Note that this example is still a fairly simple case: finding the weak limit is usually much harder, and often requires usage of very advanced tools from functional analysis. What is even worse, the weak limit in question should usually be somehow anticipated intuitively, and everything we can do is to prove (or disprove) that the anticipated result is a correct one. The main problem is the fact that the definition of weak limit is not constructive in nature. It does not give any clue about how the weak limit may be calculated. Fortunately, through some physical reasoning, the authors of this paper discovered (and proved) a constructive rule which may be used for calculating weak limit in many cases which occur in the signal theory. The rule is given through the following theorem.

Theorem 2. Let $f_\lambda(t)$ be a family of locally integrable functions parameterized by real parameter λ. Then, the weak limit of $f_\lambda(t)$ when $\lambda \to \lambda_0$, if exists, may be expressed using the formula:

$$\underset{\lambda\to\lambda_0}{\mathrm{w.lim}}\, f_\lambda(t) = \underset{\Delta t\to 0+}{\mathrm{w.lim}} \left[\frac{1}{2\Delta t} \underset{\lambda\to\lambda_0}{\mathrm{w.lim}} \int_{t-\Delta t}^{t+\Delta t} f_\lambda(\xi)\mathrm{d}\xi \right] \tag{23}$$

The weak limit may be either ordinary or generalized function, and the existence of the left side of this formula implies the existence of the right side. Additionally, under the assumption that $f_\lambda(t)$ is tempered, the existence of the right side also implies the existence of the left side. The formula is valid even when $f_\lambda(t)$ is family of generalized functions, if the definite integral in this formula is interpreted as:

$$\int_{t-\Delta t}^{t+\Delta t} f_\lambda(\xi)\mathrm{d}\xi = f_\lambda(t)*[u(t+\Delta t) - u(t-\Delta t)] \tag{24}$$

Here, the asterisk denotes the operation of convolution. Note that both sides of equation (24) coincide whenever $f_\lambda(t)$ is an ordinary function, but the right side of equation (24) is well defined even when $f_\lambda(t)$ is a generalized function because $u(t+\Delta t) - u(t-\Delta t)$ has compact support (Vladimirov, 1979; Rudin, 1973), which is generally not true for the left side. Indeed, the convolution $f(t)*g(t)$ of a generalized function $f(t)$ with a function $g(t)$ with compact support is well defined as a functional which assigns to each test function $\phi(t)$ the value $\langle f(t), g(-t)*\phi(t)\rangle$, i.e. $\langle f(t)*g(t), \phi(t)\rangle = \langle f(t), g(-t)*\phi(t)\rangle$ where the convolution $g(-t)*\phi(t)$ of two ordinary functions has the usual meaning. Such definition is sensible, because it is easy to show that under the stated condition the convolution $g(-t)*\phi(t)$ is a valid test function too, so it may be given as an argument to $f(t)$.

The proof of the Theorem 2 is given in the Appendix at the end of the paper. The proof of the fact that the existence of the right side of equation (23) implies the existence of the left side uses the Fourier transform. That is why the assumption about temperedness of $f_\lambda(t)$ is used. Otherwise, the Fourier transform cannot be applied, at least not in the space of Schwartz distributions. This assumption very probably may be omitted completely, because we did not find any counterexample where the right side of equation (23) exists and the left side of it does not exist. However, to prove such generalization, the proof must be based on tools that do not require the usage of Fourier transform. This is the topic of our current research.

At first glance, equation (23) does not bring anything useful, because it replaces one weak limit on the left side with two weak limits on the right side. However, weak limits on the right side are usually much easier to calculate, as we will see through some examples. Let us first derive the relation (4) using equation (23):

$$\begin{aligned}\underset{\omega\to\infty}{\mathrm{w.lim}}\sin\omega t &= \underset{\Delta t\to 0+}{\mathrm{w.lim}}\left[\frac{1}{2\Delta t}\underset{\omega\to\infty}{\mathrm{w.lim}}\int_{t-\Delta t}^{t+\Delta t}\sin\omega\xi\, d\xi\right] = \underset{\Delta t\to 0+}{\mathrm{w.lim}}\left[\frac{1}{2\Delta t}\underset{\omega\to\infty}{\mathrm{w.lim}}\frac{2}{\omega}\sin\omega t\sin\omega\Delta t\right]\\ &= \underset{\Delta t\to 0+}{\mathrm{w.lim}}\left[\frac{1}{2\Delta t}\lim_{\omega\to\infty}\frac{2}{\omega}\sin\omega t\sin\omega\Delta t\right] = \underset{\Delta t\to 0+}{\mathrm{w.lim}}\,0 = \lim_{\Delta t\to 0+}0 = 0\end{aligned} \tag{25}$$

In this example, both weak limits on the right side reduce to ordinary pointwise limits, because the conditions of Theorem 1 are satisfied (the functions under consideration are locally integrable and bounded).

As a second example, let us find the weak limit of sequence $f_n(t)$ given by equation (7) for $\alpha = 2$. If we apply equation (23), we obtain:

$$\begin{aligned}\underset{n\to\infty}{\mathrm{w.lim}}\,2n^2te^{-n^2t^2}u(t) &= \underset{\Delta t\to 0+}{\mathrm{w.lim}}\left[\frac{1}{2\Delta t}\underset{n\to\infty}{\mathrm{w.lim}}\int_{t-\Delta t}^{t+\Delta t}2n^2\xi e^{-n^2\xi^2}u(\xi)\mathrm{d}\xi\right]\\ &= \underset{\Delta t\to 0+}{\mathrm{w.lim}}\left[\frac{1}{2\Delta t}\underset{n\to\infty}{\mathrm{w.lim}}\int_{\max(t-\Delta t,0)}^{\max(t+\Delta t,0)}2n^2\xi e^{-n^2\xi^2}\mathrm{d}\xi\right]\\ &= \underset{\Delta t\to 0+}{\mathrm{w.lim}}\frac{1}{2\Delta t}\underset{n\to\infty}{\mathrm{w.lim}}\left\{\begin{array}{ll} e^{-n^2(t-\Delta t)^2}-e^{-n^2(t+\Delta t)^2}, & t\geq\Delta t\\ 1-e^{-n^2(t+\Delta t)^2}, & |t|<\Delta t\\ 0, & t\leq-\Delta t\end{array}\right\}\\ &= \underset{\Delta t\to 0+}{\mathrm{w.lim}}\frac{1}{2\Delta t}\lim_{n\to\infty}\left\{\begin{array}{ll} e^{-n^2(t-\Delta t)^2}-e^{-n^2(t+\Delta t)^2}, & t\geq\Delta t\\ 1-e^{-n^2(t+\Delta t)^2}, & |t|<\Delta t\\ 0, & t\leq-\Delta t\end{array}\right\}\\ &= \underset{\Delta t\to 0+}{\mathrm{w.lim}}\frac{u(t+\Delta t)-u(t-\Delta t)}{2\Delta t} = \delta(t)\end{aligned} \tag{26}$$

Here, the inner weak limit again reduces to pointwise limit (by Theorem 1), and the resulting outer weak limit is well known (weak) limit which is sometimes used in technical books even as the definition of impulse function $\delta(t)$. In fact, this limit is a generalized (weak) derivative of the step function $u(t)$.

Now, we will analyze some more sophisticated examples. Almost whole theory of the inverse Fourier transform is based on something that may be expressed as the following weak limit:

$$\underset{\varepsilon\to 0}{\mathrm{w.lim}}\frac{1}{\pi t}\sin\frac{t}{\varepsilon} = \delta(t) \tag{27}$$

This relation (in a form of ordinary limit) is often used in engineering books without any rigorous proof (Papoulis, 1962, 1977). In fact, it is not so easy to prove equation (27) rigorously. For example, the fact that all test functions have bounded variation must be used in the proof (Vladimirov, 1979; Reed and Simon, 1980; Rudin, 1973). Let us derive equation (27) using equation (23):

$$\begin{aligned}
\operatorname*{w.lim}_{\varepsilon\to 0}\frac{1}{\pi t}\sin\frac{t}{\varepsilon} &= \operatorname*{w.lim}_{\Delta t\to 0+}\left[\frac{1}{2\Delta t}\operatorname*{w.lim}_{\varepsilon\to 0}\frac{1}{\pi}\int_{t-\Delta t}^{t+\Delta t}\frac{1}{\xi}\sin\frac{\xi}{\varepsilon}\,d\xi\right]\\
&= \operatorname*{w.lim}_{\Delta t\to 0+}\left[\frac{1}{2\Delta t}\operatorname*{w.lim}_{\varepsilon\to 0}\frac{1}{\pi}\left(\mathrm{Si}\frac{t+\Delta t}{\varepsilon}-\mathrm{Si}\frac{t-\Delta t}{\varepsilon}\right)\right]\\
&= \operatorname*{w.lim}_{\Delta t\to 0+}\left[\frac{1}{2\Delta t}\lim_{\varepsilon\to 0}\frac{1}{\pi}\left(\mathrm{Si}\frac{t+\Delta t}{\varepsilon}-\mathrm{Si}\frac{t-\Delta t}{\varepsilon}\right)\right]\\
&= \operatorname*{w.lim}_{\Delta t\to 0+}\frac{u(t+\Delta t)-u(t-\Delta t)}{2\Delta t}=\delta(t)
\end{aligned} \tag{28}$$

Here, we used the boundedness of sine integral function Si t to reduce the inner weak limit to the ordinary limit. Also, the fact that Si $t\to\pm\pi/2$ for $t\to\pm\infty$ is used. Note that the resulting inner weak limit is the same as in previous example. From the proof of Theorem 2, we will see that this is not just a coincidence. Namely, the inner limit in equation (23) does not depend on particular family $f_\lambda(t)$ which converges weakly to some function $f(t)$, but only on the limit $f(t)$. This means that for each family $f_\lambda(t)$ which converges to the same weak limit $f(t)$, the inner weak limit in equation (23) will be the same (and $f(t)$ is just the weak derivative of this limit). That is why equation (23) is so useful: it reduces all weak limits that produce the same result to one unique weak limit (usually to a well-known one). Such weak limits may easily be tabulated (all of them are, in fact, weak derivatives).

Next example is even more sophisticated than previous one. Let us consider the following weak limit, which is very important in the linear frequency modulation theory (Papoulis, 1977):

$$\operatorname*{w.lim}_{\varepsilon\to 0}\frac{1}{\varepsilon\sqrt{i\pi}}e^{i(t^2/\varepsilon^2)}=\delta(t) \tag{29}$$

If we take real parts of both sides of equation (29), it may be expressed as:

$$\operatorname*{w.lim}_{\varepsilon\to 0}\frac{1}{\varepsilon\sqrt{\pi}}\sin\left(\frac{t^2}{\varepsilon^2}+\frac{\pi}{4}\right)=\delta(t) \tag{30}$$

This relation is quite far from obvious, even if we draw graphs of function under the weak limit in equation (30) for various decreasing values of ε. The following picture shows how the graph of this function looks like for one particular value of ε (Figure 2).

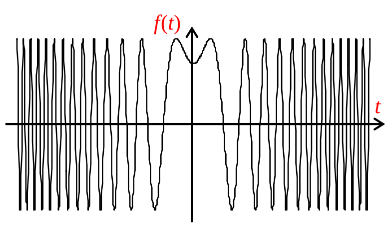

Figure 2. Function under limit in equation (30) for a particular choice of ε

As ε decreases, the oscillations become more rapid, the amplitude of oscillations increases and the central part becomes narrower. From this, it is not easy to conclude anything useful about the limit behavior. Papoulis (1977) just gives a very vague explanation and says that limit (29) is "a consequence of the oscillatory nature of the function in consideration". But, this result is not easy to anticipate, and even if someone deduces that the limit may be $\delta(t)$, it is not so easy to prove this conclusion. Let us check what equation (23) says about this limit:

$$\begin{aligned}
\underset{\varepsilon\to 0}{\mathrm{w.lim}}\frac{1}{\varepsilon\sqrt{\pi}}\sin\left(\frac{t^2}{\varepsilon^2}+\frac{\pi}{4}\right) &= \underset{\Delta t\to 0+}{\mathrm{w.lim}}\left[\frac{1}{2\Delta t}\underset{\varepsilon\to 0}{\mathrm{w.lim}}\frac{1}{\varepsilon\sqrt{\pi}}\int_{t-\Delta t}^{t+\Delta t}\sin\left(\frac{\xi^2}{\varepsilon^2}+\frac{\pi}{4}\right)d\xi\right] \\
&= \underset{\Delta t\to 0+}{\mathrm{w.lim}}\left\{\frac{1}{2\Delta t}\underset{\varepsilon\to 0}{\mathrm{w.lim}}\frac{1}{2}\mathrm{sgn}\,\varepsilon\left[\mathrm{S}\left(\frac{(t+\Delta t)\sqrt{2}}{|\varepsilon|\sqrt{\pi}}\right)-\mathrm{S}\left(\frac{(t-\Delta t)\sqrt{2}}{|\varepsilon|\sqrt{\pi}}\right)\right.\right. \\
&\left.\left.+\mathrm{C}\left(\frac{(t+\Delta t)\sqrt{2}}{|\varepsilon|\sqrt{\pi}}\right)-\mathrm{C}\left(\frac{(t-\Delta t)\sqrt{2}}{|\varepsilon|\sqrt{\pi}}\right)\right]\right\} \\
&= \underset{\Delta t\to 0+}{\mathrm{w.lim}}\left\{\frac{1}{2\Delta t}\lim_{\varepsilon\to 0}\frac{1}{2}\mathrm{sgn}\,\varepsilon\left[\mathrm{S}\left(\frac{(t+\Delta t)\sqrt{2}}{|\varepsilon|\sqrt{\pi}}\right)-\mathrm{S}\left(\frac{(t-\Delta t)\sqrt{2}}{|\varepsilon|\sqrt{\pi}}\right)\right.\right. \\
&\left.\left.+\mathrm{C}\left(\frac{(t+\Delta t)\sqrt{2}}{|\varepsilon|\sqrt{\pi}}\right)-\mathrm{C}\left(\frac{(t-\Delta t)\sqrt{2}}{|\varepsilon|\sqrt{\pi}}\right)\right]\right\} \\
&= \underset{\Delta t\to 0+}{\mathrm{w.lim}} = \frac{u(t+\Delta t)-u(t-\Delta t)}{2\Delta t} = \delta(t)
\end{aligned} \tag{31}$$

Here, $S(t)$ and $C(t)$ are Fresnel sine and cosine integrals, respectively. As we can see, the result is derived quite easily. Namely, first the boundedness of these functions is used to reduce the inner weak limit to a pointwise limit. Afterwards, the fact that $S(t) \to \pm 1/2$ and $C(t) \to \pm 1/2$ when $t \to \pm\infty$ is used to calculate the actual pointwise limit. Finally, as expected, the result is again the weak derivative of the step function.

Sometimes, the inner limit in equation (23) does not reduce to the pointwise limit. In such cases, equation (23) may be applied iteratively. For example, let us find what the weak limit of $\varepsilon t/(\varepsilon^2 + t^2)^2$ is when $\varepsilon \to 0$:

$$\begin{aligned}
\underset{\varepsilon\to 0}{\mathrm{w.lim}}\frac{\varepsilon t}{(\varepsilon^2+t^2)^2} &= \underset{\Delta t\to 0+}{\mathrm{w.lim}}\left[\frac{1}{2\Delta t}\underset{\varepsilon\to 0}{\mathrm{w.lim}}\int_{t-\Delta t}^{t+\Delta t}\frac{\varepsilon\xi}{(\varepsilon^2+\xi^2)^2}d\xi\right] \\
&= \underset{\Delta t\to 0+}{\mathrm{w.lim}}\left\{\frac{1}{2\Delta t}\underset{\varepsilon\to 0}{\mathrm{w.lim}}\frac{\varepsilon}{2}\left[\frac{1}{(t-\Delta t)^2+\varepsilon^2}-\frac{1}{(t+\Delta t)^2+\varepsilon^2}\right]\right\}
\end{aligned} \tag{32}$$

Now, the inner weak limit does not reduce to the ordinary pointwise limit. Nevertheless, we can apply equation (23) again to these weak limits. As the weak limit is linear, it is enough to calculate the following limit:

$$\underset{\varepsilon\to 0}{\mathrm{w.lim}}\frac{\varepsilon}{(t+a)^2+\varepsilon^2}=\underset{\Delta t\to 0+}{\mathrm{w.lim}}\left[\frac{1}{2\Delta t}\underset{\varepsilon\to 0}{\mathrm{w.lim}}\int_{t-\Delta t}^{t+\Delta t}\frac{\varepsilon}{(\xi+a)^2+\varepsilon^2}d\xi\right]$$
$$=\underset{\Delta t\to 0+}{\mathrm{w.lim}}\left[\frac{1}{2\Delta t}\underset{\varepsilon\to 0}{\mathrm{w.lim}}\left(\arctan\frac{t+\Delta t+a}{|\varepsilon|}-\arctan\frac{t-\Delta t+a}{|\varepsilon|}\right)\right] \quad (33)$$
$$=\underset{\Delta t\to 0+}{\mathrm{w.lim}}\left\{\frac{\pi[u(t+\Delta t+a)-u(t-\Delta t+a)]}{2\Delta t}\right\}=\pi\delta(t+a)$$

Now, the inner weak limit reduces to the pointwise limit. For its calculation, we used the fact that arc tan $t\to\pm\pi/2$ for $t\to\pm\infty$. If we insert this result back in equation (32), we can calculate the final result:

$$\underset{\varepsilon\to 0}{\mathrm{w.lim}}\frac{\varepsilon t}{(\varepsilon^2+t^2)^2}=\underset{\Delta t\to 0+}{\mathrm{w.lim}}\left\{\frac{1}{2\Delta t}\underset{\varepsilon\to 0}{\mathrm{w.lim}}\frac{\pi}{2}[\delta(t-\Delta t)-\delta(t+\Delta t)]\right\}=-\frac{\pi}{2}\delta'(t) \quad (34)$$

Here, the result is expressed using the weak derivative of impulse function (the dipole generalized function). It is possible to present much more examples of the application of Theorem 2, where other interesting singular generalized functions (other than impulse functions) are involved, like principal value of $1/t$, etc. Such examples will not be presented due to shortage of space.

4. Some applications of the weak limit in signal theory

The weak limit is a very useful concept in the analog signal theory. As already mentioned, nearly all limits in the analog signal theory are weak limits. However, the analog signal theory is full of other concepts that cannot be treated rigorously using tools from the classical calculus. Although the functional analysis offers tools that allow rigorous treatment of such concepts, the problem is that the functional analysis has its own terminology, language and abstract operator-based notation that is very indirect in nature, completely different from the usual engineering notation and even quite confusing for any non-expert in functional analysis. In this section, we will see that some of these concepts may be simply expressed rigorously through the concept of the weak limit, using the notation that is very similar to the common notation known from the ordinary calculus.

The first such concept is the concept of derivative. Analog signal theory very often says that the impulse function $\delta(t)$ is the derivative of the Heaviside step function $u(t)$. It is clear that such statement is meaningless under the usual interpretation of the derivative known from ordinary calculus. That is why functional analysis introduces the concept of so-called distributional or weak derivative. We say that the weak derivative $f'(t)$ of an ordinary or generalized function $f(t)$ is the functional that assigns to each test function $\phi(t)$ the value $-\langle f(t), \phi'(t)\rangle$. In other words, we have:

$$\langle f'(t), \phi(t)\rangle = -\langle f(t), \phi'(t)\rangle \quad (35)$$

From this definition, it seems that the weak derivative is always a generalized function. However, according to equation (22), if there exists an ordinary locally integrable function $f'(t)$ such that:

$$\int_{-\infty}^{\infty} f'(t)\phi(t)\mathrm{d}t = -\langle f(t), \phi'(t)\rangle \tag{36}$$

then $f'(t)$ is the weak derivative of $f(t)$.

It is possible to prove that the weak derivative $f'(t)$ coincides with the ordinary derivative whenever the latter one exists (Vladimirov, 1979). It is also easy to see that $\delta(t)$ is actually the weak derivative of $u(t)$. Indeed, using the partial integration it is straightforward to prove that $-\langle u(t), \phi'(t)\rangle$ is equal to $\phi(0)$, which is equal to $\langle \delta(t), \phi(t)\rangle$. However, definitions (35) and (36) are quite indirect. Although they allow to determine how $f'(t)$ acts to any test function $\phi(t)$, these definitions do not offer a way to express $f'(t)$ using known ordinary or generalized functions for a given $f(t)$. Moreover, these definitions are quite different in nature from the usual definition of the derivative. The following theorem demonstrates how the weak derivative may be expressed in a way that is quite similar to the common definition of the derivative, which also gives a method for its efficient calculation.

Theorem 3. Let $f(t)$ be an ordinary locally integrable or generalized function. Then, its weak derivative $f'(t)$ exists and may be expressed using the formula:

$$f'(t) = \underset{\Delta t \to 0}{\mathrm{w.lim}} \frac{f(t+\Delta t) - f(t)}{\Delta t} \tag{37}$$

which differs from the ordinary definition of the derivative only in the usage of the weak limit instead of the ordinary pointwise limit. When $f(t)$ is a distribution, the expression $f(t+\Delta t)$ should be interpreted using the rule $\langle f(t+\Delta t), \phi(t)\rangle = \langle f(t), \phi(t-\Delta t)\rangle$, which is true for the ordinary functions.

The proof of the Theorem 3 is given in the Appendix at the end of the paper. Although the proof is easy and straightforward, it is not possible to see this theorem and formula (37) in the common literature. The probable reason for this is the common belief that equation (37) just relates one indirect concept with another indirect concept, so equation (37) is not any more useful than the definition (35). However, this paper shows that the weak limit may be effectively calculated, so the formula (37) is in fact quite useful. Moreover, it looks much more common and much more understandable for any non-expert in functional analysis than the indirect definition (35).

The second example is even more striking than the previous one. Nearly whole analog signal theory is based on the concept of Fourier transform. However, the classical definition of the Fourier transform based on the formula:

$$\mathcal{F}\{f(t)\} = \mathrm{F}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\mathrm{d}t \tag{38}$$

works only for functions which are absolutely integrable on $(-\infty, \infty)$. Many Fourier transform pairs cannot be deduced from equation (38). One such example is given in the introductory section of the paper. In fact, it is not possible to rigorously apply equation (38) even to derive very common Fourier Transform pair $\mathcal{F}\{\mathrm{sinc}\ t\} = \pi[u(\omega+1) - u(\omega-1)]$ where $\mathrm{sinc}\ t = (\sin\ t)/t$ for $t \neq 0$ and $\mathrm{sinc}\ 0 = 0$. Namely, $f(t) = \mathrm{sinc}\ t$ is not absolutely integrable on $(-\infty, \infty)$ so the integral in equation (38) does not converge properly (it exists only as a Riemann improper integral). Moreover, it is clear that equation (38) cannot be applicable in general using the usual notion of the Lebesgue integral (nor using the concept of the Riemann improper integral). For example, well known Fourier transform pair $\mathcal{F}\{1\} = 2\pi\ \delta(t)$ cannot be deduced from equation (38), because it is not possible to get the generalized function $\delta(t)$ as a result of ordinary integration. Therefore, functional analysis introduces various more general definitions of the Fourier transform that may be applied to more general class of functions, even for generalized ones. Unfortunately, as the generality of such definitions increases, they become more and more indirect and unconstructive in nature. One of the most general definitions, which is applicable to all ordinary and generalized functions under the assumptions that they are tempered, defines the Fourier transform $\mathcal{F}\{f(t)\}$ of $f(t)$ as a purely symbolic object $\mathrm{F}(\omega)$, which acts as a functional that assigns to each test function $\phi(\omega)$ the value $\langle f(\omega), \mathcal{F}\{\phi(\omega)\}\rangle$, where $\mathcal{F}\{\phi(\omega)\}$ is defined in usual way using equation (38). In other words:

$$\langle \mathcal{F}\{f(t)\}, \phi(\omega)\rangle = \langle \mathrm{F}(\omega), \phi(\omega)\rangle = \langle f(t), \mathcal{F}\{\phi(\omega)\}\rangle \tag{39}$$

Particularly, according to equation (22), if there exists an ordinary locally integrable function $\mathrm{F}(\omega)$ such that:

$$\int_{-\infty}^{\infty} \mathrm{F}(\omega)\phi(\omega)\mathrm{d}\omega = \langle f(t), \mathcal{F}\{\phi(\omega)\}\rangle \tag{40}$$

then $\mathrm{F}(\omega)$ is the Fourier transform of $f(t)$.

It is possible to prove that such definition of the Fourier transform is sensible whenever $f(t)$ is tempered, and that it coincides with equation (38) whenever equation (38) is applicable. However, such definition is extremely indirect due to strong dependence of the operator approach, and it is almost impossible to apply this definition for actual calculation of the Fourier transform, i.e. to express the Fourier transform of some function using other common functions (ordinary or generalized), except in very simple cases. In addition, this definition is quite different from the common definition of the Fourier transform used in analog signal theory. That is why the following theorem is useful, which says that the common definition of the Fourier transform is applicable to much greater class of functions, under somewhat different interpretation of the integral in equation (38) that uses weak limits.

Theorem 4. Let $f(t)$ be an ordinary function that is locally integrable and tempered. Then, its Fourier transform exists and may be expressed using formula:

$$\mathcal{F}\{f(t)\} = \mathrm{F}(\omega) = \underset{k\to\infty}{\mathrm{w.lim}} \int_{-k}^{k} f(t)e^{-i\omega t}\mathrm{d}t \tag{41}$$

The limit in this formula is, in a sense, the weak Cauchy principal value. The similar formula is also valid for the inverse Fourier transform. It is even possible to generalize these formulae for some classes of generalized functions too, but such generalizations require the generalization of the concept of the integral. Such generalizations are out of scope of this paper. The theorem is proven in the Appendix at the end of the paper.

Note that this theorem explains easily why equation (1) is valid and why $\mathcal{F}\{1\} = 2\pi\ \delta(t)$.

5. Conclusion

This paper illustrates the concept of weak convergence, and the simple method that allows the calculation of the weak limits. The paper introduces these concepts in a natural way, which is not hard to follow even without deep knowledge of advanced mathematics. The presented concept is simple enough that it may be inserted in advanced courses of analog signal theory. Such insertion might considerably increase rigorousness of the whole signal theory as presented in engineer books.

Theorem 2 also opens some potential possibilities for implementation of calculus with generalized functions in software packages that can deal with symbolic calculus, like Wolfram Mathematica. Some research in this direction is already done by the authors, which might eventually lead to general way of representing generalized functions in software packages. Such research is out of scope of this paper.

The obvious advantage of the formula presented by Theorem 2 is its clearness and efficiency for weakly convergent sequences that arise in the signal theory. One of its main disadvantages is the fact that integration of the sequence in consideration may sometimes produce very complicated integrals, which cannot be expressed in closed form using functions with known properties. In such cases, it may be very hard to apply the formula. This problem is especially emphasized if the formula must be applied iteratively before the result is obtained. Another problem is the fact that we usually do not have a clue how many times the formula needs to be applied before we get ordinary limit instead of weak one. It is not hard to prove that this goal must be achieved in a finite number of steps if the sequence in consideration weakly converges to an ordinary function or generalized function of finite order. However, it seems that it is very hard to anticipate the necessary number of steps in advance (it depends of the order of the result, which is not known in advance). Fortunately, this number is typically very small for sequences that appear in the signal theory.

As far as the applications are concerned, it has already been shown how the concept of weak limit justifies formal differentiation and Fourier transform, but that is not the only application that comes in mind when dealing with weak limits. We already concluded that the formula proposed is crucial in weak limit calculation, which arises even more often than one may assume – but it is too often dealt with in manner of ordinary limits.

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Appendix. Proofs of stated theorems

Proof of Theorem 2. Roughly speaking, Theorem 2 says that in order to calculate a weak limit of a sequence, we can calculate the weak derivative of the weak limit of an integrated sequence. To prove the theorem, we need to recall some auxiliary but quite advanced results from functional analysis, especially from the theory of convolutions of generalized functions (Vladimirov, 1979; Rudin, 1973; Schwartz, 1965). First, we recall that the mapping $f(t) \rightarrow f(t)*h(t)$ is continuous mapping from the space of generalized functions $\mathcal{D}^*(\mathbb{R})$ into itself in respect to weak limit whenever $h(t)$ has the compact support (i.e. whenever it is time limited). This means that under the stated conditions, the convolution $f_\lambda(t)*h(t)$ converges weakly to $f(t)*h(t)$ whenever $f_\lambda(t)$ converges weakly to $f(t)$ (Vladimirov, 1979; Rudin, 1973). Assume now that weak limit of $f_\lambda(t)$ when $\lambda \rightarrow \lambda_0$ exists and that it is equal to $f(t)$. Since the function $u(t + \Delta t) - u(t - \Delta t)$ has compact support, we can write:

$$\begin{aligned} \underset{\lambda\rightarrow\lambda_0}{\mathrm{w.lim}} \int_{t-\Delta t}^{t+\Delta t} f_\lambda(\xi)\mathrm{d}\xi &= \underset{\lambda\rightarrow\lambda_0}{\mathrm{w.lim}}[f_\lambda(t)*[u(t+\Delta t)-u(t-\Delta t)]] \\ &= \left[\underset{\lambda\rightarrow\lambda_0}{\mathrm{w.lim}} f_\lambda(t)\right]*[u(t+\Delta t)-u(t-\Delta t)] = f(t)*[u(t+\Delta t)-u(t-\Delta t)] \end{aligned} \tag{A1}$$

Note that the convolution in above formula is well defined, because $u(t + \Delta t) - u(t - \Delta t)$ has the compact support. Now, we need to recall that the mapping $g(t) \rightarrow g(t)*h(t)$ is continuous mapping with respect to weak limit even when $h(t)$ is not time limited if we restrict the domain of this mapping to the space of functions $g(t)$ whose supports lay inside one fixed interval $(-t_{max}, t_{max})$ which does not depend on the particular function $g(t)$ (Vladimirov, 1979). This is true even if $h(t)$ is generalized function (i.e. a distribution). The family of functions $g_{\Delta t}(t) = [u(t + \Delta t) - u(t - \Delta t)]/(2\,\Delta t)$ parameterized in Δt where $\Delta t \in (0, t_{max})$ satisfy this condition. Then, if we recall that $\delta(t)$ is a weak derivative of $u(t)$, and that $\delta(t)$ is the identity element for the convolution (Vladimirov, 1979; Reed and Simon, 1980; Rudin, 1973; Schwartz, 1965), we can write:

$$\begin{aligned} \underset{\Delta t\rightarrow 0+}{\mathrm{w.lim}}\left[\frac{1}{2\Delta t}\underset{\lambda\rightarrow\lambda_0}{\mathrm{w.lim}}\int_{t-\Delta t}^{t+\Delta t} f_\lambda(\xi)\mathrm{d}\xi\right] &= \underset{\Delta t\rightarrow 0+}{\mathrm{w.lim}}\left[\frac{1}{2\Delta t}f(t)*[u(t+\Delta t)-u(t-\Delta t)]\right] \\ &= \underset{\Delta t\rightarrow 0+}{\mathrm{w.lim}}\left[\frac{u(t+\Delta t)-u(t-\Delta t)}{2\Delta t}*f(t)\right] \\ &= \underset{\Delta t\rightarrow 0+}{\mathrm{w.lim}}\left[\frac{u(t+\Delta t)-u(t-\Delta t)}{2\Delta t}\right]*f(t) = \delta(t)*f(t) = f(t) \end{aligned} \tag{A2}$$

Therefore, assuming that the left side of equation (23) exists, we prove that the right side of equation (23) exists too, and that they are equal. The proof of auxiliary facts used is far from trivial, and may be found in Vladimirov (1979).

Now, we need to prove that the existence of the right side of equation (23) implies the existence of the left side of equation (23). In other words, we need to prove that if the right side of equation (23) exists, then $f_\lambda(t)$ is weakly convergent when $\lambda \rightarrow \lambda_0$. We will introduce the family $w_{\lambda,\Delta t}(t)$ of ordinary or generalized functions parameterized in both λ and Δt as:

$$w_{\lambda,\Delta t}(t) = \frac{1}{2\Delta t}\int_{t-\Delta t}^{t+\Delta t} f_\lambda(\xi)\mathrm{d}\xi = \frac{1}{2\Delta t} f_\lambda(t)*[u(t+\Delta t) - u(t-\Delta t)] \tag{A3}$$

Under the assumption that the right side of equation (23) exists, $w_{\lambda,\Delta t}(t)$ obviously must converge weakly when $\lambda \rightarrow \lambda_0$ at least for values of Δt which belong to some right neighborhood of zero, i.e. which belong to some interval $(0, \tau)$. Applying the Fourier transform to equation (A3) and using the well-known Convolution Theorem, we get:

$$\begin{aligned} \mathrm{W}_{\lambda,\Delta t}(\omega) &= \mathcal{F}\{w_{\lambda,\Delta t}(t)\} = \mathcal{F}\left\{\frac{1}{2\Delta t} f_\lambda(t) * [u(t+\Delta t) - u(t-\Delta t)]\right\} \\ &= \frac{1}{2\Delta t}\mathcal{F}\{f_\lambda(t)\}\cdot\mathcal{F}\{u(t+\Delta t) - u(t-\Delta t)\} = \mathrm{F}_\lambda(\omega)\mathrm{sinc}\,\omega\Delta t \end{aligned} \tag{A4}$$

where $\mathrm{W}_{\lambda,\Delta t}(\omega)$ and $\mathrm{F}_\lambda(\omega)$ are the Fourier transforms of $w_{\lambda,\Delta t}(t)$ and $f_\lambda(t)$, respectively. As both the Fourier transform and the inverse Fourier transform are continuous mappings, a family of tempered distributions is weakly convergent if and only if the family of their Fourier transforms is weakly convergent. Particularly, if $w_{\lambda,\Delta t}(t)$ is weakly convergent for some value of Δt when $\lambda \to \lambda_0$, the same is true for $\mathrm{W}_{\lambda,\Delta t}(\omega)$. Let see which conclusions we can derive about the behavior of $\mathrm{F}_\lambda(\omega)$ under such assumption. We will first prove that for a given $\mathrm{W}_{\lambda,\Delta t}(\omega)$, there always exist infinitely many distributions $\mathrm{F}_\lambda(\omega)$ that satisfy equation (A4) in the distributional sense. Following the lines of a proof about the distributional solutions of equations like $(t - t_0) f(t) = g(t)$ given in Vladimirov (1979), we will show that one such distribution is $\overset{*}{\mathrm{F}}_{\lambda,\Delta t}(\omega)$ defined with the rule:

$$\left\langle \overset{*}{\mathrm{F}}_{\lambda,\Delta t}(\omega), \phi(\omega) \right\rangle = \left\langle \mathrm{W}_{\lambda,\Delta t}(\omega), \frac{1}{\mathrm{sinc}\,\omega\Delta t}\left[\phi(\omega) - \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \phi\left(\frac{k\pi}{\Delta t}\right)\eta\left(\omega - \frac{k\pi}{\Delta t}\right)\right]\right\rangle \tag{A5}$$

where $\eta(\omega)$ is an arbitrary but fixed test function equal to 1 in some neighborhood of $\omega = 0$. The definition (A5) is meaningful, because if $\phi(\omega)$ is a test function, it is easy to show that the function:

$$\psi(\omega) = \frac{1}{\mathrm{sinc}\,\omega\Delta t}\left[\phi(\omega) - \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \phi\left(\frac{k\pi}{\Delta t}\right)\eta\left(\omega - \frac{k\pi}{\Delta t}\right)\right] \tag{A6}$$

is also a test function. Namely, although it seems that $\psi(\omega)$ is singular whenever $\mathrm{sinc}\,\omega\Delta t = 0$, all such singularities are removable ones. Furthermore, as the product of a distribution $f(\omega)$ with a smooth function $\sigma(\omega)$ is defined using the rule $\langle f(\omega)\,\sigma(\omega), \phi(\omega)\rangle = \langle f(\omega), \phi(\omega)\,\sigma(\omega)\rangle$, we have:

$$\begin{aligned} \left\langle \overset{*}{\mathrm{F}}_{\lambda,\Delta t}(\omega)\mathrm{sinc}\omega\Delta t, \phi(\omega)\right\rangle &= \left\langle \overset{*}{\mathrm{F}}_{\lambda,\Delta_t}(\omega), \phi(\omega)\mathrm{sinc}\omega\Delta t\right\rangle \\ &= \left\langle \mathrm{W}_{\lambda,\Delta t}(\omega), \frac{1}{\mathrm{sinc}\omega\Delta t}\left[\phi(\omega)\mathrm{sinc}\omega\Delta t \right.\right. \\ &\quad \left.\left. - \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \phi\left(\frac{k\pi}{\Delta t}\right)\eta\left(\omega - \frac{k\pi}{\Delta t}\right)\mathrm{sinc}k\pi\right\rangle\right] = \left\langle \mathrm{W}_{\lambda,\Delta t}(\omega), \phi(\omega)\right\rangle \end{aligned} \tag{A7}$$

Therefore, $\mathrm{F}_\lambda(\omega) = \overset{*}{\mathrm{F}}_{\lambda,\Delta t}(\omega)$ really satisfies equation (A4).

Suppose now that there is another solution $\mathrm{F}_\lambda(\omega)$ different from $\overset{*}{\mathrm{F}}_{\lambda,\Delta t}(\omega)$ that also satisfies equation (A4). Then, the difference $\mathrm{F}_\lambda(\omega) - \overset{*}{\mathrm{F}}_{\lambda,\Delta t}(\omega)$ must satisfy the equation:

$$[\mathrm{F}_\lambda(\omega) - \overset{*}{\mathrm{F}}_{\lambda,\Delta t}(\omega)]\mathrm{sinc}\,\omega\Delta t = 0 \tag{A8}$$

From equation (A6) we have:

$$\phi(\omega) = \psi(\omega)\operatorname{sinc}\omega\Delta t + \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \phi\left(\frac{k\pi}{\Delta t}\right)\eta\left(\omega - \frac{k\pi}{\Delta t}\right) \tag{A9}$$

Therefore:

$$\begin{aligned}
\left\langle F_\lambda(\omega) - F^*_{\lambda,\Delta t}(\omega), \phi(\omega)\right\rangle &= \left\langle F_\lambda(\omega) - F^*_{\lambda,\Delta t}(\omega), \psi(\omega)\operatorname{sinc}\omega\Delta t + \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \phi\left(\frac{k\pi}{\Delta t}\right)\eta\left(\omega - \frac{k\pi}{\Delta t}\right)\right\rangle \\
&= \left\langle F_\lambda(\omega) - F^*_{\lambda,\Delta t}(\omega), \psi(\omega)\operatorname{sinc}\omega\Delta t\right\rangle \\
&\quad + \left\langle F_\lambda(\omega) - F^*_{\lambda,\Delta t}(\omega), \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \phi\left(\frac{k\pi}{\Delta t}\right)\eta\left(\omega - \frac{k\pi}{\Delta t}\right)\right\rangle \\
&= \left\langle [F_\lambda(\omega) - F^*_{\lambda,\Delta t}(\omega)]\operatorname{sinc}\omega\Delta t, \psi(\omega)\right\rangle + \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \phi\left(\frac{k\pi}{\Delta t}\right) \\
&\quad \times \left\langle F_\lambda(\omega) - F^*_{\lambda,\Delta t}(\omega), \eta\left(\omega - \frac{k\pi}{\Delta t}\right)\right\rangle \\
&= \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \phi\left(\frac{k\pi}{\Delta t}\right)c_k(\lambda,\Delta t) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} c_k(\lambda,\Delta t)\left\langle \delta\left(\omega - \frac{k\pi}{\Delta t}\right), \phi(\omega)\right\rangle \\
&= \left\langle \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} c_k(\lambda,\Delta t)\delta\left(\omega - \frac{k\pi}{\Delta t}\right), \phi(\omega)\right\rangle
\end{aligned} \tag{A10}$$

Here we introduced notation:

$$c_k(\lambda,\Delta t) = \left\langle F_\lambda(\omega) - F^*_{\lambda,\Delta t}(\omega), \eta\left(\omega - \frac{k\pi}{\Delta t}\right)\right\rangle \tag{A11}$$

The values $c_k(\lambda, \Delta t)$ are constants (which may depend on λ and Δt), because $\eta(\omega)$ is a fixed function. From equation (A10) we can conclude that:

$$F_\lambda(\omega) - F^*_{\lambda,\Delta t}(\omega) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} c_k(\lambda, \Delta t)\delta\left(\omega - \frac{k\pi}{\Delta t}\right) \tag{A12}$$

Therefore, any eventual solution of equation (A4) for $F_\lambda(\omega)$ when $W_{\lambda,\Delta t}(\omega)$ is given must have the form:

$$F_\lambda(\omega) = F^*_{\lambda,\Delta t}(\omega) + \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} c_k(\lambda, \Delta t)\delta\left(\omega - \frac{k\pi}{\Delta t}\right) \tag{A13}$$

where $F^*_{\lambda,\Delta t}(\omega)$ is defined by equation (A5). Moreover, the straightforward calculation shows that equation (A13) satisfies equation (A4) for arbitrary values of $c_k(\lambda, \Delta t)$. This means that equation (A13) is the general solution of equation (A4) for $F_\lambda(\omega)$ when $W_{\lambda,\Delta t}(\omega)$ is given. Note that in general the right side of equation (A13) may depend of Δt. But if equation (A13) represents the actual $F_\lambda(\omega)$, the Fourier transform of $f_\lambda(t)$, Δt must be canceled somehow inside the right side of equation (A13), as the actual $F_\lambda(\omega)$ does not depend of Δt.

We saw earlier that $W_{\lambda,\Delta t}(\omega)$ converges weakly when $\lambda \to \lambda_0$ for $\Delta t \in (0, \tau)$. It is easy to see that same is true for $F^*_{\lambda,\Delta t}(\omega)$. Indeed, the weak convergence of $W_{\lambda,\Delta t}(\omega)$ when $\lambda \to \lambda_0$ for fixed Δt means that the sequence of numbers $\langle W_{\lambda,\Delta t}(\omega), \psi(\omega)\rangle$ converges when $\lambda \to \lambda_0$ for each test function $\psi(\omega)$. But $\langle F^*_{\lambda,\Delta t}(\omega), \varphi(\omega)\rangle$ is equal to $\langle W_{\lambda,\Delta t}(\omega), \psi(\omega)\rangle$ where $\psi(\omega)$ is given by equation (A6). This means that the sequence of numbers $\langle F^*_{\lambda,\Delta t}(\omega), \varphi(\omega)\rangle$ converges when $\lambda \to \lambda_0$ for each test function $\varphi(\omega)$. In other words, $F^*_{\lambda,\Delta t}(\omega)$ converges weakly when $\lambda \to \lambda_0$.

Note that although $F^*_{\lambda,\Delta t}(\omega)$ for fixed $\Delta t \in (0, \tau)$ converges weakly when $\lambda \to \lambda_0$, the right side of equation (A13) in general may be weakly divergent. This will be the case whenever some of values $c_k(\lambda, \Delta t)$ behave wildly when $\lambda \to \lambda_0$. Now, we will show that this is not possible if equation (A13) really represents the actual $F_\lambda(\omega)$. The key point is the independence of the right side of equation (A14) on Δt. Indeed, for any two values $\Delta t = \Delta t'$ and $\Delta t = \Delta t''$ from $(0, \tau)$ the right sides of equation (A13) must be equal. Therefore, we have:

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} c_k(\lambda, \Delta t'')\delta\left(\omega - \frac{k\pi}{\Delta t''}\right) - \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} c_k(\lambda, \Delta t')\delta\left(\omega - \frac{k\pi}{\Delta t'}\right) = F^*_{\lambda,\Delta t'}(\omega) - F^*_{\lambda,\Delta t''}(\omega) \tag{A14}$$

As the right side of equation (A14) is weakly convergent when $\lambda \to \lambda_0$, the same must be true for the left side. This is possible only if the both sums at the left side of equation (A13) are either simultaneously weakly convergent or weakly divergent when $\lambda \to \lambda_0$. However, as the both sums are in fact the impulse trains, the latter scenario is possible only if there is a cancellation of some impulses in the first sum with some impulses in the second sum. Such cancellation requires that $\Delta t'$ and $\Delta t''$ have the greatest common divisor. Although such cancellation is possible for some particular values of $\Delta t'$ and $\Delta t''$, it is not possible for arbitrary values of $\Delta t'$ and $\Delta t''$. But the left side of equation (A14) must be weakly convergent when $\lambda \to \lambda_0$ for arbitrary values of $\Delta t'$ and $\Delta t''$. Therefore, both sums in equation (A14) must be weakly convergent when $\lambda \to \lambda_0$. In other words, the sum in equation (A13) must be weakly convergent when $\lambda \to \lambda_0$ for arbitrary value Δt from $(0, \tau)$. The consequence is that $F_\lambda(\omega)$ also converges weakly when $\lambda \to \lambda_0$, because we saw that both components of equation (A13) are weakly convergent. As the Inverse Fourier Transform is a continuous mapping, the same is true for $f_\lambda(t)$, so the weak limit of $f_\lambda(t)$ when $\lambda \to \lambda_0$ exists. The proof is now completed. □

Proof of Theorem 3. It is a well-known fact that the weak derivative of an ordinary locally integrable function or a generalized function always exists (Vladimirov, 1979). As $f(t)$ may be expressed as a linear and continuous functional using equation (22) (if it is an ordinary locally integrable function), or it is already a linear and continuous functional (if it is a generalized function), it is possible to write the following chain of equalities:

$$\begin{aligned}
\lim_{\Delta t\to 0}\left\langle \frac{f(t+\Delta t)-f(t)}{\Delta t}, \phi(t)\right\rangle &= \lim_{\Delta t\to 0}\left\langle f(t+\Delta t)-f(t), \frac{\phi(t)}{\Delta t}\right\rangle \\
&= \lim_{\Delta t\to 0}\left\langle f(t+\Delta t), \frac{\phi(t)}{\Delta t}\right\rangle - \lim_{\Delta t\to 0}\left\langle f(t), \frac{\phi(t)}{\Delta t}\right\rangle \\
&= \lim_{\Delta t\to 0}\left\langle f(t), \frac{\phi(t-\Delta t)}{\Delta t}\right\rangle - \lim_{\Delta t\to 0}\left\langle f(t), \frac{\phi(t)}{\Delta t}\right\rangle \\
&= \lim_{\Delta t\to 0}\left\langle f(t), \frac{\phi(t-\Delta t)-\phi(t)}{\Delta t}\right\rangle \\
&= \left\langle f(t), \lim_{\Delta t\to 0}\frac{\phi(t-\Delta t)-\phi(t)}{\Delta t}\right\rangle = \langle f(t), -\phi'(t)\rangle \\
&= -\langle f(t), \phi(t)\rangle = \langle f'(t), \phi(t)\rangle
\end{aligned} \tag{A15}$$

This means that equation (37) holds, so the proof is completed. □

Proof of Theorem 4. We recall some facts about appropriate functions spaces known from functional analysis (Vladimirov, 1979; Reed and Simon, 1980; Rudin, 1973). At first, the integral in equation (41) is the Fourier transform of the function $f(t)\, u(k-|t|)$. Indeed, under the stated conditions, $f(t)\, u(k-|t|)$ vanishes out of the interval $(-k, k)$, so it is absolutely integrable. Therefore, equation (38) is applicable, and the integral in equation (38) reduces to the form given in equation (41). Now, the theorem is simply a consequence of the facts that the Fourier transform is a continuous mapping from the space of tempered distributions $S^*(\mathbb{R})$ to itself, and that $f(t)\, u(k-|t|)$ weakly converges to $f(t)$ when $k \to \infty$ (even in the sense of the topology of $S^*(\mathbb{R})$). Therefore, we have:

$$\begin{aligned}
\mathcal{F}\{f(t)\} &= \mathcal{F}\left\{ \underset{k\to\infty}{\mathrm{w.lim}}\, f(t)u(k-|t|)\right\} = \underset{k\to\infty}{\mathrm{w.lim}}\, \mathcal{F}\{f(t)u(k-|t|)\} \\
&= \underset{k\to\infty}{\mathrm{w.lim}} \int_{-k}^{k} f(t)e^{-i\omega t}\mathrm{d}t
\end{aligned} \tag{A16}$$

This proves the theorem. The proof of auxiliary facts used may be found in Vladimirov (1979), Reed and Simon (1980) and Rudin (1973). □

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